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On the relationship between convex bodies related to correlation experiments with dichotomic observables

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Abstract

In this paper we explore further the connections between convex bodies related to quantum correlation experiments with dichotomic variables and related bodies studied in combinatorial optimization, especially cut polyhedra. Such a relationship was established in Avis *et al* (2005 *J. Phys. A: Math. Gen.* **38** 10971–87) with respect to Bell inequalities. We show that several well-known bodies related to cut polyhedra are equivalent to bodies such as those defined by Tsirelson (1993 *Hadronic J. Suppl.* **8** 329–45) to represent hidden deterministic behaviours, quantum behaviours and no-signalling behaviours. Among other things, our results allow a unique representation of these bodies, give a necessary condition for vertices of the no-signalling polytope, and give a method for bounding the quantum violation of Bell inequalities by means of a body that contains the set of quantum behaviours. Optimization over this latter body may be performed efficiently by semidefinite programming. In the second part of the paper we apply these results to the study of classical correlation functions. We provide a complete list of tight inequalities for the two party case with (m, n) dichotomic observables when $m = 4, n = 4$ and when $\min\{m, n\} \leq 3$, and give a new general family of correlation inequalities.

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1. The convex sets arising from the classical, quantum and no-signaling correlation experiments

Our terminology for bodies related to quantum correlations follows closely that of Tsirelson [34], whilst for cut polyhedra we follow that of Deza and Laurent [13]. We consider the following (*quantum*) *correlation experiment*. Suppose that two parties, say Alice and Bob, share a quantum mixed state ρ , or a non-negative Hermitian operator ρ on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ with $\text{tr } \rho = 1$. Here \mathcal{H}_A and \mathcal{H}_B are Hilbert spaces representing the subsystems owned by Alice and Bob, respectively. Alice has $m \pm 1$ -valued observables A_1, \dots, A_m in the space \mathcal{H}_A , i.e. Hermitian operators A_i on \mathcal{H}_A whose eigenvalues are within $[-1, 1]$. Similarly Bob has $n \pm 1$ -valued observables B_1, \dots, B_n in \mathcal{H}_B . Alice and Bob measure one observable each, say A_i and B_j . By repeating this process with different choices of i, j , we collect the probabilities $q_{ab|ij}$ with which A_i measures to a and B_j to b simultaneously under the condition that Alice measures A_i and Bob B_j for $1 \leq i \leq m, 1 \leq j \leq n, a, b \in \{\pm 1\}$. Using ρ, A_i, B_j , these probabilities are calculated as $q_{ab|ij} = \text{tr}(\rho(\frac{1+aA_i}{2} \otimes \frac{1+bB_j}{2}))$. The result of such a correlation experiment can be seen as a $4mn$ -dimensional real vector $\mathbf{q} \in \mathbb{R}^{4mn}$. A correlation experiment is said to be *classical* if the state ρ is separable.

All possible results of (quantum) correlation experiments satisfy the following three conditions. The *non-negativity condition* is $q_{ab|ij} \geq 0$ for $1 \leq i \leq m, 1 \leq j \leq n, a, b \in \{\pm 1\}$. The *normalization condition* is $\sum_{a,b \in \{\pm 1\}} q_{ab|ij} = 1$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. The *no-signaling condition* means that there exist marginal probabilities $q_{a|i}^A = \sum_{b \in \{\pm 1\}} q_{ab|ij}$ and $q_{b|j}^B = \sum_{a \in \{\pm 1\}} q_{ab|ij}$ which are independent of j and i , respectively. A vector $\mathbf{q} \in \mathbb{R}^{4mn}$ satisfying these conditions is called a *behaviour*, and a vector which is a possible result of a quantum (resp. classical) correlation experiment is called a *quantum* (resp. *hidden deterministic*) *behaviour* [34]. We denote the sets of all behaviours, of all quantum behaviours and of all hidden deterministic behaviours by $X_B(m, n)$, $X_{QB}(m, n)$ and $X_{HDB}(m, n)$, respectively. Whereas Tsirelson [34] defines them in the general case where each observable has an arbitrary number of outcomes, we consider the special case that each observable has two possible outcomes.

Froissart [14] shows that $X_{HDB}(m, n)$ is a $(mn + m + n)$ -dimensional convex polytope which has 2^{m+n} vertices corresponding to the cases where observables A_i and B_j are fixed constant $+1$ or -1 , or in other words, $A_1, \dots, A_m, B_1, \dots, B_n \in \{\pm 1\}$.

A linear inequality on $q_{ab|ij}$ which is satisfied for all possible results of classical correlation experiments, or for all the points in $X_{HDB}(m, n)$, is called a *Bell inequality* (for (m, n) settings). However, this is cumbersome for certain purposes because, as is pointed out by Froissart [14], adding any linear combination of the normalization and no-signaling conditions to an inequality gives apparently different representations of essentially the same inequality. To avoid this, we consider a full-dimensional polytope isomorphic to $X_{HDB}(m, n)$, which will be described shortly. The set $X_{QB}(m, n)$ is a $(mn + m + n)$ -dimensional convex, bounded, closed set [34]. Recently, Barnett, Linden, Massar, Pironio, Popescu and Roberts [5] studied the vertices of the polytope consisting of all behaviours with two observables per party. We call $X_B(m, n)$ the *no-signaling polytope* following [5].

Due to the normalization and no-signaling conditions, a behaviour \mathbf{q} is completely specified by the values of $p_{A_i B_j} = q_{-1, -1|ij}$, the marginal probabilities $p_{A_i} = q_{-1|i}^A$ and $p_{B_j} = q_{-1|j}^B$. We consider \mathbf{p} as an $(mn + m + n)$ -dimensional vector in the vector space $\mathbb{R}^{V_{m,n} \cup E_{m,n}}$, where $V_{m,n} = \{A_1, \dots, A_m, B_1, \dots, B_n\}$ and $E_{m,n} = \{A_i B_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ are the node set and the edge set, respectively, of the complete bipartite graph $K_{m,n}$. We denote the vector $\mathbf{q} \in \mathbb{R}^{4mn}$ corresponding to a given vector $\mathbf{p} \in \mathbb{R}^{V_{m,n} \cup E_{m,n}}$ by $\iota(\mathbf{p})$.

Formally, ι is a one-to-one affine mapping from $\mathbb{R}^{V_{m,n} \cup E_{m,n}}$ into \mathbb{R}^{4mn} defined by $q_{-1,-1|ij} = p_{A_i B_j}$, $q_{-1,+1|ij} = p_{A_i} - p_{A_i B_j}$, $q_{+1,-1|ij} = p_{B_j} - p_{A_i B_j}$, $q_{+1,+1|ij} = 1 - p_{A_i} - p_{B_j} + p_{A_i B_j}$.

This means that we can consider the convex bodies (convex, bounded, closed, full-dimensional sets) $\iota^{-1}(X_B(m, n))$, $\iota^{-1}(X_{QB}(m, n))$ and $\iota^{-1}(X_{HDB}(m, n))$ instead of $X_B(m, n)$, $X_{QB}(m, n)$ and $X_{HDB}(m, n)$, respectively. Especially, $\iota^{-1}(X_{HDB}(m, n))$ is exactly identical to the *correlation polytope* [32] of the complete bipartite graph $K_{m,n}$, which we denote by $\text{COR}^\square(K_{m,n})$ following Deza and Laurent [13]. The correlation polytope of a graph is introduced by Pitowsky [32] (see also [30]) to describe the possible results of classical correlation experiments in a broader sense than our use of the term, and our case corresponds to the correlation polytope of the complete bipartite graph $K_{m,n}$. The correlation polytope has been also studied in the context of combinatorial optimization under the name ‘boolean quadric polytope’ in relation to unconstrained quadratic 0-1 programming [27] (see section 5.1 of [13]). We denote $\iota^{-1}(X_{QB}(m, n))$ by $\mathcal{Q}(m, n)$. We refer to $\mathcal{Q}(m, n)$ as the *quantum correlation set*.

Bell inequalities can be written by using p_{A_i} , p_{B_j} , $p_{A_i B_j}$ instead of $q_{ab|ij}$. Using vectors \mathbf{p} , a Bell inequality is exactly a linear inequality satisfied for all the points in $\text{COR}^\square(K_{m,n})$. This avoids the problem stated above because $\text{COR}^\square(K_{m,n})$ is full-dimensional and representation of an inequality is unique up to positive scaling. From now on, we represent Bell inequalities in terms of \mathbf{p} .

Aside from trivial examples such as $p_{A_1 B_1} \geq 0$, $p_{A_1 B_1} - p_{B_1} \leq 0$ or $p_{A_1} + p_{B_1} - p_{A_1 B_1} \leq 1$, a non-trivial example with two observables per party is the famous Clauser–Horne–Shimony–Holt (CHSH) inequality [9]:

$$-p_{A_1} - p_{B_1} + p_{A_1 B_1} + p_{A_1 B_2} + p_{A_2 B_1} - p_{A_2 B_2} \leq 0. \tag{1}$$

Any inequality that can be described as a sum of two different Bell inequalities is trivially a Bell inequality, and such a Bell inequality is said to be *redundant*. A Bell inequality is said to be *tight* if not redundant. From the central theorem in the theory of convex polytopes (see, e.g., section 1.1 of [37]), for any fixed m and n , there are finitely many tight Bell inequalities, which give a unique minimum representation of the correlation polytope $\text{COR}^\square(K_{m,n})$ by inequalities.

The most famous example of a linear inequality valid for the quantum correlation set is Tsirelson’s inequality [8] stating that the maximum violation of the CHSH inequality (1) in the quantum case is $\sqrt{2} - 1$. This maximum is achieved by using a maximally entangled pure state in the two-qubit system and suitable observables. Pitowsky [31] also considers this set. We note that Tsirelson [8] states an exact characterization of the set $X_{QB}(2, 2)$ by a system of algebraic equations and inequalities on finitely many variables with quantifiers.⁵

Figure 1 gives an overview of most of the results we will see in this paper. The two leftmost columns labelled as ‘ \mathbb{R}^{4mn} ’ and ‘ $\mathbb{R}^{V_{m,n} \cup E_{m,n}}$ ’ depict the relationship explained so far except for the relations involving $\text{RCMet}(K_{m,n})$, the rooted correlation semimetric polytope introduced in the following section. The ‘complexity’ column refers to the computational complexity of testing membership in the given body. The rest of the figure will be explained in the following sections.

2. The no-signaling polytope and the rooted correlation semimetric polytope

We will prove that the no-signaling polytope, if represented in terms of vectors $\mathbf{p} \in \mathbb{R}^{V_{m,n} \cup E_{m,n}}$ instead of the vectors $\mathbf{q} \in \mathbb{R}^{4mn}$, is identical to a convex polytope which arises in

⁵ Except for $M_{QB}(m, n)$, which is equal to $\mathcal{E}(K_{m,n})$ by corollary 2.

	\mathbb{R}^{4mn}	\hookrightarrow	$\mathbb{R}^{V_{m,n} \cup E_{m,n}}$	$\xrightarrow{\varphi}$	$\mathbb{R}^{\nabla E_{m,n}}$	$\xrightarrow{\pi \text{ (see Prop.3)}}$	$\mathbb{R}^{E_{m,n}}$	Complexity
	\cup		\cup		\cup		\cup	
No-signaling	$X_B(m, n)$	$\stackrel{\text{Th.1}}{\cong}$	$\text{RCMet}(\mathbb{K}_{m,n})$	\cong	$\text{RMet}(\nabla \mathbb{K}_{m,n})$	$\stackrel{\text{Prop.4}}{\rightarrow}$	$[-1, 1]^{E_{m,n}}$	Easy
	\cup		\cup	$\mathcal{E}(\nabla \mathbb{K}_{m,n}) \cap \text{RMet}(\nabla \mathbb{K}_{m,n}) \rightarrow$	\cup		$\mathcal{E}(\mathbb{K}_{m,n})$	Tractable
				\cup Th.6			\parallel Cor.2	
Quantum	$X_{\text{QB}}(m, n)$	\cong	$\mathcal{Q}(m, n)$	\cong	$\mathcal{Q}_{\text{Cut}}(m, n)$	\rightarrow	$M_{\text{QB}}(m, n)$	Unknown ¹
	\cup		\cup	\cup	\cup		\cup	
Classical	$X_{\text{HDB}}(m, n)$	\cong	$\text{COR}^\square(\mathbb{K}_{m,n})$	\cong	$\text{Cut}(\nabla \mathbb{K}_{m,n})$	\rightarrow	$\text{Cut}(\mathbb{K}_{m,n})$	Hard

Figure 1. Relationship among various convex sets discussed in the paper. The injective mapping ι is defined in section 1. The isomorphism φ is the covariance mapping stated in section 3. The projection π is the standard projection from $\mathbb{R}^{\nabla E_{m,n}}$ to $\mathbb{R}^{E_{m,n}}$. Proper inclusions are valid for $m, n \geq 2$.

the combinatorial optimization. The *rooted correlation semimetric polytope* of a graph $G = (V, E)$ is the convex polytope $\text{RCMet}(G)$ in $\mathbb{R}^{V \cup E}$ defined by a system of inequalities $p_{uv} \geq 0, p_u - p_{uv} \geq 0, 1 - p_u - p_v + p_{uv} \geq 0$ for $uv \in E$ [27]. Padberg [27] studies this polytope as a natural linear relaxation of the correlation polytope and investigates the relationship between them. The name ‘rooted correlation semimetric polytope’, used in Deza and Laurent [13], comes from its relation to the cut polytope explained in the next section. We note that $\text{RCMet}(\mathbb{K}_n)$ appears in Pitowsky [29].

Theorem 1. *The no-signaling polytope $X_B(m, n)$ satisfies $X_B(m, n) = \iota(\text{RCMet}(\mathbb{K}_{m,n}))$, where $\text{RCMet}(\mathbb{K}_{m,n})$ denotes the rooted correlation semimetric polytope of the complete bipartite graph $\mathbb{K}_{m,n}$.*

Proof. Since every point in $X_B(m, n)$ satisfies the normalization and the no-signaling conditions, the set $X_B(m, n)$ is contained in the image $\iota(\mathbb{R}^{V_{m,n} \cup E_{m,n}})$. Let $\mathbf{p} \in \mathbb{R}^{V_{m,n} \cup E_{m,n}}$ and $\mathbf{q} = \iota(\mathbf{p})$. From the definition of ι , the following logical equivalences hold: $q_{-1,-1|ij} \geq 0 \iff p_{A_i B_j} \geq 0, q_{-1,+1|ij} \geq 0 \iff p_{A_i} - p_{A_i B_j} \geq 0, q_{+1,-1|ij} \geq 0 \iff p_{B_j} - p_{A_i B_j} \geq 0, q_{+1,+1|ij} \geq 0 \iff 1 - p_{A_i} - p_{B_j} + p_{A_i B_j} \geq 0$. This means that $\mathbf{q} \in X_B(m, n)$ if and only if $\mathbf{p} \in \text{RCMet}(\mathbb{K}_{m,n})$, which implies $X_B(m, n) = \iota(\text{RCMet}(\mathbb{K}_{m,n}))$. \square

The vertices of the rooted correlation semimetric polytope are studied by Padberg [27].

Theorem 2 ([27]). *The coordinates of the vertices of the rooted correlation semimetric polytope $\text{RCMet}(G)$ are in $\{0, 1/2, 1\}$.*

Corollary 1. *The coordinates of the vertices of $X_B(m, n)$ are in $\{0, 1/2, 1\}$.*

Proof. Immediate from theorems 1 and 2 and the definition of ι . \square

In a related work [5], Barrett, Linden, Massar, Pironio, Popescu and Roberts investigate the vertices of the no-signaling polytope with two k -outcome observables per party.

3. The covariance mapping and the cut polytope

The correlation polytope $\text{COR}^\square(\mathbb{K}_{m,n})$ is isomorphic to the cut polytope of a certain graph, which is the suspension graph of $\mathbb{K}_{m,n}$ and has a natural physical interpretation. The *suspension graph* $\nabla \mathbb{K}_{m,n}$ of $\mathbb{K}_{m,n}$ is obtained from $\mathbb{K}_{m,n}$ by adding one new node X which is adjacent to all the other $m + n$ nodes. The graph $\nabla \mathbb{K}_{m,n} = (\nabla V_{m,n}, \nabla E_{m,n})$ has $1 + m + n$ nodes $\nabla V_{m,n} = \{X, A_1, \dots, A_m, B_1, \dots, B_n\}$ and $m + n + mn$ edges $\nabla E_{m,n} = \{XA_i \mid 1 \leq i \leq m\}$

$\cup \{XB_j \mid 1 \leq j \leq n\} \cup \{A_i B_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Here we denote the node set and the edge set of $\nabla K_{m,n}$ by $\nabla V_{m,n}$ and $\nabla E_{m,n}$, slightly abusing the notation. For any vector $c \in \{\pm 1\}^{\nabla V_{m,n}}$, the *cut vector* of $\nabla K_{m,n}$ defined by c is the vector $x \in \mathbb{R}^{\nabla E_{m,n}}$ such that $x_{uv} = c_u c_v$ for $uv \in \nabla E_{m,n}$. The convex hull of all the cut vectors of $\nabla K_{m,n}$ is called the *cut polytope* of $\nabla K_{m,n}$ and denoted by $\text{Cut}(\nabla K_{m,n})$. Research on the cut polytope has a long and rich history and many results are known; see Deza and Laurent [13]. We note that in [13] and many other papers in combinatorics, the cut polytope is defined in the terms of 0/1 cut vectors instead of ± 1 cut vectors, which we use here for better correspondence to the ± 1 -valued observables. All the results on the cut polytope can be stated both in the ± 1 terminology and in the 0/1 terminology.

The correlation polytope and the cut polytope are related via the covariance mapping [13, section 5.2]. The correlation polytope $\text{COR}^\square(K_{m,n})$ is isomorphic to the cut polytope $\text{Cut}(\nabla K_{m,n})$ via a linear isomorphism, called the *covariance mapping* φ , which maps $p \in \mathbb{R}^{V_{m,n} \cup E_{m,n}}$ to $x \in \mathbb{R}^{\nabla E_{m,n}}$ defined by $x_{XA_i} = 1 - 2p_{A_i}$, $x_{XB_j} = 1 - 2p_{B_j}$ and $x_{A_i B_j} = 1 - 2p_{A_i} - 2p_{B_j} + 4p_{A_i B_j}$.

In the classical and quantum cases, the coordinates of the vector x have a natural physical interpretation related to the observables $A_1, \dots, A_m, B_1, \dots, B_n$ as stated in the following proposition.

Proposition 1. *Let a vector $p \in \mathbb{R}^{V_{m,n} \cup E_{m,n}}$ be as defined in section 1, and let $x = \varphi(p)$. Then $x_{XA_i} = \langle A_i \rangle$, $x_{XB_j} = \langle B_j \rangle$ and $x_{A_i B_j} = \langle A_i B_j \rangle$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, where $\langle \cdot \rangle$ denotes the expected value.*

Proof. The equation $x_{XA_i} = \langle A_i \rangle$ is proved as follows: $\langle A_i \rangle = (+1) \cdot (1 - p_{A_i}) + (-1) \cdot p_{A_i} = 1 - 2p_{A_i} = x_{XA_i}$. The equations $\langle B_j \rangle = x_{XB_j}$ and $\langle A_i B_j \rangle = x_{A_i B_j}$ can be verified similarly. \square

The image of $\text{RCMet}(K_{m,n})$ under the covariance mapping φ is the *rooted semimetric polytope* of $\nabla K_{m,n}$ pointed at the node X [13, section 27.2], and is denoted by $\text{RMet}(\nabla K_{m,n})$.

Proposition 2. *The polytope $\text{RMet}(\nabla K_{m,n}) = \varphi(\text{RCMet}(K_{m,n}))$ is defined by inequalities $x_{XA_i} + x_{XB_j} + x_{A_i B_j} \geq -1$, $-x_{XA_i} - x_{XB_j} + x_{A_i B_j} \geq -1$, $x_{XA_i} - x_{XB_j} - x_{A_i B_j} \geq -1$ and $-x_{XA_i} + x_{XB_j} - x_{A_i B_j} \geq -1$ for $1 \leq i \leq m, 1 \leq j \leq n$.*

We denote the image of $\mathcal{Q}(m, n)$ under the covariance mapping φ by $\mathcal{Q}_{\text{Cut}}(m, n) = \varphi(\mathcal{Q}(m, n))$.

4. Correlation functions

The *correlation function* $x_{A_i B_j}$ is the expected value $\langle A_i B_j \rangle$ of the product of an observable by Alice and another observable by Bob. The correlation functions $x_{A_i B_j}$ for all i, j form an mn -dimensional vector $x' \in \mathbb{R}^{E_{m,n}}$. Tsirelson [8, 35, 34] gives a detailed study on the sets of correlation functions which are possible in classical and quantum correlation experiments.

Clearly a correlation function $x_{A_i B_j}$ takes a value in the range $[-1, 1]$. The value $(1 - x_{A_i B_j})/2$ in the range $[0, 1]$ is the probability of the exclusive OR of the events that A_i measures to -1 and that B_j to -1 under the condition that Alice measures A_i and Bob B_j . We note that if we replace the ‘exclusive OR’ by ‘AND’, we obtain the probability with which both A_i and B_j measure to -1 under the same condition, resulting in another convex polytope explored by Froissart [14].

From proposition 1, the vector x' is the projection of x defined in section 3 to $\mathbb{R}^{E_{m,n}}$ by the standard projection $\pi : \mathbb{R}^{\nabla E_{m,n}} \rightarrow \mathbb{R}^{E_{m,n}}$, giving the following proposition.

Proposition 3. Let $\pi : \mathbb{R}^{\nabla E_{m,n}} \rightarrow \mathbb{R}^{E_{m,n}}$ be the standard projection.

- (i) The vectors of correlation functions which are possible in classical correlation experiments form the cut polytope $\text{Cut}(\mathbf{K}_{m,n})$ of the complete bipartite graph $\mathbf{K}_{m,n}$. Here the cut polytope $\text{Cut}(\mathbf{K}_{m,n})$ is a convex polytope in $\mathbb{R}^{E_{m,n}}$ defined in the same way as $\text{Cut}(\nabla \mathbf{K}_{m,n})$, replacing $\nabla E_{m,n}$ by $E_{m,n}$.
- (ii) The vectors of correlation functions which are possible in quantum correlation experiments form a convex body $\pi(\mathcal{Q}_{\text{Cut}}(m, n))$.
- (iii) All the vectors of correlation functions which can arise from behaviours form a convex polytope $\pi(\text{RMet}(\nabla \mathbf{K}_{m,n}))$.

Proof. Immediate from proposition 1, with theorem 1 for (iii). \square

Similar to a Bell inequality, a *correlation inequality* (for (m, n) settings) is a linear inequality on correlation functions which is satisfied for all possible results of classical correlation experiments. The CHSH inequality (1) can be also written as a correlation inequality:

$$x_{A_1 B_1} + x_{A_1 B_2} + x_{A_2 B_1} - x_{A_2 B_2} \leq 2. \quad (2)$$

An N -party version of the set of classical correlation functions with two observables per party is studied by Werner and Wolf [36], and it turns out to be a crosspolytope, a convex polytope with a surprisingly simple structure compared to the complicated structure of the correlation polytope. One might expect that there is also a simple characterization of the correlation inequalities for the case with an arbitrary number of observables per party. In the two-party case, this leads to an analysis of a projection of the correlation polytope $\text{COR}^\square(\mathbf{K}_{m,n})$ and the cut polytope $\text{Cut}(\nabla \mathbf{K}_{m,n})$, which is the cut polytope $\text{Cut}(\mathbf{K}_{m,n})$ of the bipartite graph $\mathbf{K}_{m,n}$. However, as shown in [1, 30], such a characterization is unlikely, since membership testing in these polyhedra is NP-complete.

A correlation inequality for (m, n) settings is said to be *redundant* if it is a sum of two correlation inequalities for (m, n) settings which are not the scalings of it, and *tight* if not redundant. Tight correlation inequalities are exactly the facet-inducing inequalities of the corresponding polytope $\text{Cut}(\mathbf{K}_{m,n})$.

There are some ‘obvious’ symmetries acting on Bell inequalities, and they act also on correlation inequalities [36]. They are combinations of one or more of the following basic symmetries: (i) party exchange, (ii) observable exchange and (iii) relabelling of outcomes (these terms are coined by Masanes [25]). Two correlation inequalities are said to be *equivalent* if one of them can be transformed to the other by applying one or more of these basic symmetries. The cut polytope admits two basic symmetries called permutation and switching [13, sections 26.2, 26.3]. As is the case of Bell inequalities and $\text{Cut}(\nabla \mathbf{K}_{m,n})$ discussed in [2], the basic symmetries acting on correlation inequalities correspond exactly to the basic symmetries of $\text{Cut}(\mathbf{K}_{m,n})$.

In the no-signaling case, the corresponding polytope becomes the hypercube.

Proposition 4. The set $\pi(\text{RMet}(\nabla \mathbf{K}_{m,n}))$, which is the set of vectors in $\mathbb{R}^{E_{m,n}}$ realizable as the correlation function arising from behaviours, is equal to the hypercube $[-1, 1]^{E_{m,n}}$.

Proof. It is trivial that $\pi(\text{RMet}(\nabla \mathbf{K}_{m,n})) \subseteq [-1, 1]^{E_{m,n}}$.

To prove the converse, let $\mathbf{x}' \in [-1, 1]^{E_{m,n}}$. We define $\mathbf{x} \in \mathbb{R}^{\nabla E_{m,n}}$ by $x_{A_i B_j} = x'_{A_i B_j}$ and $x_{X A_i} = x_{X B_j} = 0$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then $\mathbf{x} \in \text{RMet}(\nabla \mathbf{K}_{m,n})$ and $\pi(\mathbf{x}) = \mathbf{x}'$, which implies $\pi(\text{RMet}(\nabla \mathbf{K}_{m,n})) \supseteq [-1, 1]^{E_{m,n}}$. \square

The implication of this theorem is that all correlations between the observables A_i, B_j are possible under the no-signaling condition alone.

In the quantum case, Tsirelson’s theorem [8] (see [35] for a proof) gives a beautiful characterization of the quantum bound of correlation functions.

Theorem 3 ([35]). *Let m, n be positive integers. On a real vector $\mathbf{x} \in \mathbb{R}^{E_{m,n}}$, the following assertions are equivalent.*

- (i) $\mathbf{x} \in \mathcal{Q}_{\text{Cut}}(m, n)$. In other words, there exist Hilbert spaces \mathcal{H}_A and \mathcal{H}_B , a mixed state ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ and Hermitian operators A_1, \dots, A_m on \mathcal{H}_A and B_1, \dots, B_n on \mathcal{H}_B with eigenvalues in $[-1, 1]$, such that $x_{A_i B_j} = \text{tr}[\rho(A_i \otimes B_j)]$ for $1 \leq i \leq m, 1 \leq j \leq n$.
- (ii) The same as (i) with the following additional conditions: (a) \mathcal{H}_A and \mathcal{H}_B have finite dimensions d_A and d_B , respectively, and $d_A \leq 2^{\lceil m/2 \rceil}, d_B \leq 2^{\lceil n/2 \rceil}$. (b) $A_i^2 = B_j^2 = I$ and $\text{tr}[\rho(A_i \otimes I)] = \text{tr}[\rho(I \otimes B_j)] = 0$ for $1 \leq i \leq m, 1 \leq j \leq n$. (c) Anticommutators $A_{i_1} A_{i_2} + A_{i_2} A_{i_1}$ for $1 \leq i_1 < i_2 \leq m$ and $B_{j_1} B_{j_2} + B_{j_2} B_{j_1}$ for $1 \leq j_1 < j_2 \leq n$ are scalar, that is, proportional to I .
- (iii) There exist $m + n$ unit vectors $\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{v}_1, \dots, \mathbf{v}_n$ in the vector space \mathbb{R}^{m+n} such that $x_{A_i B_j} = \mathbf{u}_i \cdot \mathbf{v}_j$.

In combinatorial optimization, vectors whose elements are defined as inner products of unit vectors, as in condition (iii), are well studied. They form a set called the *elliptope* [13, section 28.4]. The elliptope $\mathcal{E}(G)$ of a graph $G = (V, E)$ with $n = |V|$ nodes is the convex body consisting of vectors $\mathbf{x} \in \mathbb{R}^E$ such that there exists a unit vector \mathbf{u}_i in \mathbb{R}^n for each node $i \in V$ satisfying $x_{ij} = \mathbf{u}_i \cdot \mathbf{u}_j$. In particular, the elliptope $\mathcal{E}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ is the set of the vector $\mathbf{x} \in \mathbb{R}^{E_{m,n}}$ satisfying condition (iii) of theorem 3.

Corollary 2. $\pi(\mathcal{Q}_{\text{Cut}}(m, n)) = \mathcal{E}(K_{m,n})$.

It is well known that the elliptope can also be characterized by using non-negative definite matrices called Gram matrices (see, e.g., section 28.4.1 of [13]). The following theorem is the characterization of $\mathcal{E}(G)$ in this form.

Theorem 4. *Let $G = (V, E)$ be a graph with $|V| = n$ nodes labelled as $1, \dots, n$. A vector $\mathbf{x} \in \mathbb{R}^E$ satisfies $\mathbf{x} \in \mathcal{E}(G)$ if and only if there is an $n \times n$ real symmetric non-negative definite matrix $H = (h_{ij})$ such that $h_{ij} = x_{ij}$ for all $ij \in E$ and $h_{ii} = 1$ for all $1 \leq i \leq n$.*

Linear functions can be optimized efficiently over the elliptope $\mathcal{E}(G)$ by using semidefinite programming (see, e.g., [6]; this optimization is the heart of the Goemans–Williamson approximation algorithm for the maximum cut problem [19]). By efficiently we mean up to some error ε in time polynomial in the input size and $\log(1/\varepsilon)$. A similar statement is true about membership testing in $\mathcal{E}(G)$. This fact combined with corollary 2 implies that the maximum violation of given correlation inequalities in the quantum case can be computed efficiently, as pointed out by Cleve, Høyer, Toner and Watrous [10]. In addition, Grothendieck’s inequality [22], stating that the elliptope is not much larger than the cut polytope for bipartite graphs, gives an upper bound of the violation of correlation inequalities [10, 35]. Besides, as is pointed out by Tsirelson [34], Grothendieck [20] proves that for a vector $\mathbf{x} \in \mathbb{R}^{E_{m,n}}$ to belong to $\mathcal{E}(K_{m,n})$, it is necessary that the vector $\mathbf{y} \in \mathbb{R}^{E_{m,n}}$ defined by $y_{A_i B_j} = (2/\pi) \arcsin x_{A_i B_j}$ belongs to $\text{Cut}(K_{m,n})$. Tsirelson conjectures there that this condition is also sufficient for $m = n = 2$. This condition is known under the name *cut condition* [13, section 31.3.1] in combinatorial optimization, and necessary for a vector to belong to $\mathcal{E}(G)$ with any graph G (where the cut condition for $\mathcal{E}(G)$ is defined analogously). Laurent [23] proves that the cut condition for $\mathcal{E}(G)$ is sufficient if and only if G has no K_4 -minor. According to this, the cut

condition for $\mathcal{E}(\mathbb{K}_{m,n})$ is sufficient if and only if $\min\{m, n\} \leq 2$, and Tsirelson's conjecture is settled affirmatively.

Pitowsky [31] considers the intersection of $\mathcal{Q}(m, n)$ with the subspace U of $\mathbb{R}^{V_{m,n} \cup E_{m,n}}$ defined by $m+n$ equations $p_{A_i} = 1/2$ and $p_{B_j} = 1/2$, and proves the following theorem as a corollary of the equivalence of conditions (i) and (ii) in theorem 3.⁶

Theorem 5. *An affine mapping from $\mathbb{R}^{V_{m,n} \cup E_{m,n}}$ to itself which maps \mathbf{p} to \mathbf{p}' defined by $p'_{A_i} = p'_{B_j} = 1/2$ and $p'_{A_i B_j} = p_{A_i B_j} - \frac{1}{2}p_{A_i} - \frac{1}{2}p_{B_j} + \frac{1}{2}$ maps $\mathcal{Q}(m, n)$ onto $\mathcal{Q}(m, n) \cap U$.*

The affine mapping in theorem 5 can be explained by using the cut polytope and the covariance mapping. The vectors $\varphi(\mathbf{p}')$ are equal to $\varphi(\mathbf{p})$ in the coordinates corresponding to $E_{m,n}$, and zero in the other coordinates. Therefore, theorem 5 is equivalent to stating that, for any vector \mathbf{x} in $\varphi(\mathcal{Q}(m, n)) = \mathcal{Q}_{\text{Cut}}(m, n)$, the vector \mathbf{y} obtained from \mathbf{x} by replacing the $m+n$ coordinates $x_{X_{A_i}}$ and $x_{X_{B_j}}$ by zero also belongs to $\mathcal{Q}_{\text{Cut}}(m, n)$. This property follows from the symmetry of the set $\mathcal{Q}_{\text{Cut}}(m, n)$, and the same holds also for $\text{Cut}(\nabla\mathbb{K}_{m,n})$ and $\text{RMet}(\nabla\mathbb{K}_{m,n})$.

5. Implication of Tsirelson's theorem on the quantum correlation set

In this section we describe a body that contains the quantum correlation set $\mathcal{Q}_{\text{Cut}}(m, n)$, and give some related applications. The body is described in the following theorem, the proof of which is based on corollary 2 of Tsirelson's theorem.

Theorem 6. $\mathcal{Q}_{\text{Cut}}(m, n) \subseteq \mathcal{E}(\nabla\mathbb{K}_{m,n}) \cap \text{RMet}(\nabla\mathbb{K}_{m,n})$.

Proof. $\mathcal{Q}_{\text{Cut}}(m, n) = \varphi(\mathcal{Q}(m, n)) \subseteq \text{RMet}(\nabla\mathbb{K}_{m,n})$ follows from $\mathcal{Q}(m, n) \subseteq \text{RCMet}(\mathbb{K}_{m,n})$ and $\varphi(\text{RCMet}(\mathbb{K}_{m,n})) = \text{RMet}(\nabla\mathbb{K}_{m,n})$.

To prove $\mathcal{Q}_{\text{Cut}}(m, n) \subseteq \mathcal{E}(\nabla\mathbb{K}_{m,n})$, let $\mathbf{x} \in \mathcal{Q}_{\text{Cut}}(m, n)$. Then there exist Hilbert spaces \mathcal{H}_A and \mathcal{H}_B , a quantum state ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$, m Hermitian operators A_1, \dots, A_m on \mathcal{H}_A and n Hermitian operators B_1, \dots, B_n on \mathcal{H}_B such that $x_{X_{A_i}} = \langle A_i \rangle$, $x_{X_{B_j}} = \langle B_j \rangle$, $x_{A_i B_j} = \langle A_i B_j \rangle$. We add a Hermitian operator $A_{m+1} = I$ and $B_{n+1} = I$ and consider the bipartite case $\mathbb{K}_{m+1, n+1}$. Define $\mathbf{y} \in \mathbb{R}^{E_{m+1, n+1}}$ by $y_{A_i B_j} = \langle A_i B_j \rangle$. Then for $1 \leq i \leq m$ and $1 \leq j \leq n$, $y_{A_i B_j} = x_{A_i B_j}$, $y_{A_i B_{n+1}} = x_{X_{A_i}}$, $y_{A_{m+1} B_j} = x_{X_{B_j}}$ and $y_{A_{m+1} B_{n+1}} = 1$. Corollary 2 guarantees $\mathbf{y} \in \mathcal{E}(\mathbb{K}_{m+1, n+1})$. This means that there are unit vectors $\mathbf{u}_1, \dots, \mathbf{u}_{m+1}, \mathbf{v}_1, \dots, \mathbf{v}_{n+1}$ in the vector space \mathbb{R}^{m+n+2} such that $y_{A_i B_j} = \mathbf{u}_i \cdot \mathbf{v}_j$ for $1 \leq i \leq m+1$ and $1 \leq j \leq n+1$. From $y_{A_{m+1} B_{n+1}} = 1$, we have $\mathbf{u}_{m+1} = \mathbf{v}_{n+1}$. Note that the $m+n+2$ vectors \mathbf{u}_i and \mathbf{v}_j lie in an $(m+n+1)$ -dimensional subspace of \mathbb{R}^{m+n+2} because there are at most $m+n+1$ distinct vectors among them. This means $\mathbf{x} \in \mathcal{E}(\nabla\mathbb{K}_{m,n})$. \square

Remark 1. Given corollary 2, one may expect that $\mathcal{Q}_{\text{Cut}}(m, n) = \mathcal{E}(\nabla\mathbb{K}_{m,n})$, but this can be easily disproved as follows. Since a vector $\mathbf{x} \in \mathbb{R}^{V_{m,n}}$ defined by $x_{X_{A_i}} = x_{X_{B_j}} = x_{A_i B_j} = -1/4$ for all $1 \leq i \leq m$, $1 \leq j \leq n$ lies in $\mathcal{E}(\nabla\mathbb{K}_{m,n}) \setminus \text{RMet}(\nabla\mathbb{K}_{m,n})$, we have $\mathcal{E}(\nabla\mathbb{K}_{m,n}) \not\subseteq \text{RMet}(\nabla\mathbb{K}_{m,n})$ and therefore $\mathcal{E}(\nabla\mathbb{K}_{m,n}) \not\subseteq \mathcal{Q}_{\text{Cut}}(m, n)$ for any $m, n \geq 1$. On the other hand, we do not know whether the inclusion $\mathcal{Q}_{\text{Cut}}(m, n) \subseteq \mathcal{E}(\nabla\mathbb{K}_{m,n}) \cap \text{RMet}(\nabla\mathbb{K}_{m,n})$ is proper or not.

Since linear functions can be optimized efficiently over $\mathcal{E}(\nabla\mathbb{K}_{m,n})$ by the interior-point method, theorem 6 can be used to give an upper bound of the maximum quantum violation of any Bell inequality.

⁶ Strictly speaking, Pitowsky [31] considers the subset $\mathcal{Q}_{\text{finite}}(m, n)$ of $\mathcal{Q}(m, n)$ consisting of quantum behaviours which can be realized with ρ finite-dimensional quantum states. In [31] it is proved that the affine mapping in theorem 5 maps $\mathcal{Q}_{\text{finite}}(m, n)$ onto $\mathcal{Q}_{\text{finite}}(m, n) \cap U$. Since Tsirelson's theorem is valid also for infinite-dimensional quantum systems, the same proof is valid for infinite-dimensional systems.

For example, Collins and Gisin [11] show that for a tight Bell inequality for (3, 3) settings called I_{3322} [11],

$$f_{3322} = -x_{XA_1} - x_{XA_2} + x_{XB_1} + x_{XB_2} + x_{A_1B_1} + x_{A_1B_2} + x_{A_1B_3} + x_{A_2B_1} \\ + x_{A_2B_2} - x_{A_2B_3} + x_{A_3B_1} - x_{A_3B_2} \leq 4,$$

one can achieve $f_{3322} = 9/2$ in $\mathcal{Q}_{\text{Cut}}(3, 3)$ by using the maximally entangled state with $\dim \mathcal{H}_A = \dim \mathcal{H}_B = 2$ and appropriate observables. Using the SDPA package [15] for semidefinite programming, we calculated the maximum of f_{3322} over $\mathcal{E}(\nabla K_{3,3}) \cap \text{RMet}(\nabla K_{3,3})$ as 5.4641, whose exact value seems to be $2(\sqrt{3} + 1)$, with unit vectors corresponding to the nodes $X, A_1, A_2, A_3, B_1, B_2, B_3$ of $K_{3,3}$ which lie in a four-dimensional space and their coordinates in \mathbb{R}^4 are

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_1 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 - \sqrt{3} \\ 0 \\ 2 \\ \sqrt{3} + 1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 - \sqrt{3} \\ 0 \\ -2 \\ \sqrt{3} + 1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \\ \mathbf{v}_1 = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3} - 1 \\ 2 \\ 0 \\ \sqrt{3} + 1 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3} - 1 \\ -2 \\ 0 \\ \sqrt{3} + 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Like Tsirelson's theorem, theorem 6 can be used to test quantum mechanics itself. As Froissart pointed out [14], finding the violation of Bell inequalities does not prove or disprove quantum mechanics. However, we do not know if theorem 6 provides a stronger test than Tsirelson's theorem, since we do not know if there is $\mathbf{x} \in \text{RMet}(\nabla K_{m,n}) \setminus \mathcal{E}(\nabla K_{m,n})$ such that $\pi(\mathbf{x}) \in \mathcal{E}(K_{m,n})$.

Now we consider the set $X_{\text{QB}}(m, n)$. It is clear from the definition of $\mathcal{Q}_{\text{Cut}}(m, n)$ that theorems 1 and 6 imply that $X_{\text{QB}}(m, n) \subseteq \iota(\varphi^{-1}(\mathcal{E}(\nabla K_{m,n}))) \cap X_{\text{B}}(m, n)$. The following theorem replaces the right-hand side of this inclusion with a simpler set. Here we compare $X_{\text{QB}}(m, n)$ with the elliptope $\mathcal{E}(K_{2m,2n})$ of the complete bipartite graph $K_{2m,2n} = (V_{2m,2n}, E_{2m,2n})$ as follows. We label the nodes of $K_{2m,2n}$ by $A_{a,i}$ (for $a \in \{\pm 1\}$ and $1 \leq i \leq m$) and $B_{b,j}$ (for $b \in \{\pm 1\}$ and $1 \leq j \leq n$), and we identify the $4mn$ -dimensional vector space $\mathbb{R}^{E_{2m,2n}}$ with \mathbb{R}^{4mn} introduced in section 1 by mapping an edge $A_{a,i}B_{b,j}$ to the coordinate $q_{ab|ij}$.

Theorem 7. $X_{\text{QB}}(m, n) \subseteq \mathcal{E}(K_{2m,2n}) \cap X_{\text{B}}(m, n)$.

Proof. $X_{\text{QB}}(m, n) \subseteq X_{\text{B}}(m, n)$ is obvious. $X_{\text{QB}}(m, n) \subseteq \mathcal{E}(K_{2m,2n})$ is immediate from theorem 6 and the following lemma. \square

Lemma 1. $\iota(\varphi^{-1}(\mathcal{E}(\nabla K_{m,n}))) \subseteq \mathcal{E}(K_{2m,2n})$.

Proof. Let $\mathbf{x} \in \mathcal{E}(\nabla K_{m,n})$ and $\mathbf{q} = \iota(\varphi^{-1}(\mathbf{x}))$. We prove $\mathbf{q} \in \mathcal{E}(K_{2m,2n})$.

By definition of the elliptope $\mathcal{E}(\nabla K_{m,n})$, there exist unit vectors \mathbf{u}_i for $1 \leq i \leq m$, \mathbf{v}_j for $1 \leq j \leq n$, and \mathbf{w} in \mathbb{R}^{1+m+n} such that $x_{XA_i} = \mathbf{w} \cdot \mathbf{u}_i$, $x_{XB_j} = \mathbf{w} \cdot \mathbf{v}_j$ and $x_{A_iB_j} = \mathbf{u}_i \cdot \mathbf{v}_j$. By a simple calculation, \mathbf{q} is written as $q_{ab|ij} = \mathbf{u}'_{a,i} \cdot \mathbf{v}'_{b,j}$ using vectors $\mathbf{u}'_{a,i} = (\mathbf{w} + a\mathbf{u}_i)/2$ and $\mathbf{v}'_{b,j} = (\mathbf{w} + b\mathbf{v}_j)/2$ in \mathbb{R}^{1+m+n} whose lengths are at most 1. By using a well-known technique, we can replace $\mathbf{u}'_{a,i}$ and $\mathbf{v}'_{b,j}$ by unit vectors $\mathbf{u}''_{a,i}$ and $\mathbf{v}''_{b,j}$ in \mathbb{R}^{2m+2n} preserving their inner products: $q_{ab|ij} = \mathbf{u}''_{a,i} \cdot \mathbf{v}''_{b,j}$. Namely, we add some coordinates to the space to convert the vectors $\mathbf{u}'_{a,i}$ and $\mathbf{v}'_{b,j}$ to unit vectors in a higher-dimensional space, and then we restrict the space to the subspace spanned by the $2m + 2n$ vectors. This proves that $\mathbf{q} \in \mathcal{E}(K_{2m,2n})$. \square

However, the usefulness of theorem 7 is yet to be investigated. For example, theorem 7 provides little information about $X_{QB}(3, 3)$ since even $X_B(3, 3) \subsetneq \text{Cut}(K_{6,6})$ holds, which is proved by enumerating the vertices of $X_B(3, 3)$ by using cdd [16] (cddlib 0.94b).

6. Correlation inequalities and their tightness

In this section, we apply results on facet-inducing inequalities of the cut polytope to the case of $\text{Cut}(K_{m,n})$ to see the implications of the relationship between the set of classical correlation functions and the cut polytope $\text{Cut}(K_{m,n})$.

6.1. Trivial and cycle inequalities as correlation inequalities

Barahona and Mahjoub [4] study two classes of inequalities valid for the cut polytope $\text{Cut}(G)$ of an arbitrary graph $G = (V, E)$, and characterize which of these inequalities are facet inducing.

For $uv \in E$, *trivial inequalities* are $x_{uv} \leq 1$ and $x_{uv} \geq -1$, which are switching equivalent to each other. They are facet inducing for $\text{Cut}(G)$ if and only if the edge uv does not belong to any triangle in G . For a cycle $C = \{u_1u_2, u_2u_3, \dots, u_{l-1}u_l, u_lu_1\} \subseteq E$ and a subset $F \subseteq C$ with $|F|$ odd, a *cycle inequality* is $-\sum_{e \in F} x_e + \sum_{e \in C \setminus F} x_e \leq |C| - 2$. An example is inequality (2), where $C = \{A_1B_1, A_2B_1, A_2B_2, A_1B_2\}$ and $F = \{A_2B_2\}$. Cycle inequalities with a common cycle C and different subsets F are switching equivalent to one another. They are facet inducing for $\text{Cut}(G)$ if and only if the cycle C is a chordless cycle, i.e., no two nodes in C form an edge in G other than an edge in C . Note that the case of inequality (2) satisfies this condition.

The following theorem follows from this.

Theorem 8. *The trivial inequalities $x_{A_iB_j} \leq 1$ and $x_{A_iB_j} \geq -1$ for $1 \leq i \leq m, 1 \leq j \leq n$ and the CHSH inequalities $x_{A_{i_1}B_{j_1}} + x_{A_{i_1}B_{j_2}} + x_{A_{i_2}B_{j_1}} - x_{A_{i_2}B_{j_2}} \leq 2$ and $x_{A_{i_1}B_{j_1}} - x_{A_{i_1}B_{j_2}} - x_{A_{i_2}B_{j_1}} - x_{A_{i_2}B_{j_2}} \leq 2$ for $1 \leq i_1, i_2 \leq m, 1 \leq j_1, j_2 \leq n, i_1 \neq i_2, j_1 \neq j_2$ are tight correlation inequalities.*

6.2. All correlation inequalities with (4, 4) settings

Gisin [18] listed all correlation inequalities with a small number of settings per party that involve small integer coefficients. As a result, with (2, 2) or (3, s) settings with $s = 2, 3, 4$, the CHSH inequality seems the only non-trivial correlation inequality, while with (4, 4) or more settings other correlation inequalities exist. He found two facet-inducing correlation inequalities with (4, 4) settings.

Let $\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} a_{A_iB_j} x_{A_iB_j} \leq a_0$ be a correlation inequality. We denote this inequality by extracting the coefficients on the left-hand side as

$$\begin{pmatrix} & (B_1) & \dots & (B_s) \\ (A_1) & a_{A_1B_1} & \dots & a_{A_1B_s} \\ \vdots & \vdots & & \vdots \\ (A_r) & a_{A_rB_1} & \dots & a_{A_rB_s} \end{pmatrix} \leq a_0.$$

For example, the CHSH inequality (2) is written as

$$\begin{pmatrix} & (B_1) & (B_2) \\ (A_1) & 1 & 1 \\ (A_2) & 1 & -1 \end{pmatrix} \leq 2.$$

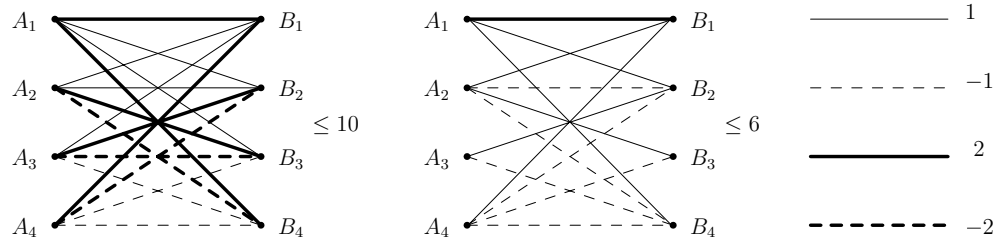


Figure 2. The tight correlation inequalities (3) and (4).

Using this notation, the inequalities found by Gisin are

$$\begin{pmatrix} & (B_1) & (B_2) & (B_3) & (B_4) \\ (A_1) & 2 & 1 & 1 & 2 \\ (A_2) & 1 & 1 & 2 & -2 \\ (A_3) & 1 & 2 & -2 & -1 \\ (A_4) & 2 & -2 & -1 & -1 \end{pmatrix} \leq 10, \tag{3}$$

$$\begin{pmatrix} & (B_1) & (B_2) & (B_3) & (B_4) \\ (A_1) & 2 & 1 & 0 & 1 \\ (A_2) & 1 & -1 & 1 & -1 \\ (A_3) & 0 & 1 & 0 & -1 \\ (A_4) & 1 & -1 & -1 & -1 \end{pmatrix} \leq 6. \tag{4}$$

They are shown in figure 2. Moreover, Gisin showed that these two are only inequivalent facet correlation inequalities with (4, 4) settings and absolute values of coefficients at most two. He also found that there is exactly one facet correlation inequality that involves exactly (m, n) settings with coefficients 0, ±1 for (m, n) = (4, 5), (5, 5).

To see whether there are any other tight correlation inequalities if we do not restrict the range of coefficients, we enumerated the facet inequalities of Cut(K_{4,4}) by using cdd [16] (cddlib 0.94b). After that, we filtered out equivalent correlation inequalities by using nauty [26] (nauty 2.2). We obtained the following result.

Theorem 9. *The trivial inequality $x_{A_1 B_1} \leq 1$, the CHSH inequality (2), and the two inequalities (3) and (4) found by Gisin are all the inequivalent facet inequalities of Cut(K_{4,4}).*

6.3. Zero-lifting of correlation inequalities

If a Bell inequality for (m, n) settings is tight, then it is also a tight Bell inequality for (m', n'), where 1 ≤ m ≤ m' and 1 ≤ n ≤ n'. This can be proved directly [28], or as a corollary [2] of the zero-lifting theorem for the cut polytope of a general graph [12]. Here we prove that the same holds for correlation inequalities.

Theorem 10. *Let 1 ≤ m ≤ m' and 1 ≤ n ≤ n'. A tight correlation inequality for (m, n) settings is also a tight correlation inequality for (m', n') settings.*

Proof. It suffices if we prove that an inequality $\mathbf{a}^T \mathbf{x} \leq a_0$ which is facet inducing for Cut(K_{m,n}) is also facet inducing for Cut(K_{m',n'}).

If the inequality involves only one coordinate, then it is necessarily equivalent to the trivial inequality $x_{A_1 B_1} \leq 1$, and it is facet inducing for Cut(K_{m',n'}) for any m', n' by theorem 8.

If the inequality involves more than one coordinate, then the theorem follows by applying the zero-lifting theorem for the cut polytope of a general graph [12] repeatedly, in the same way as in the case [2] of Bell inequalities. \square

6.4. Correlation inequalities for $(3, n)$ settings

From Gisin's observation that the CHSH inequality is the only non-trivial correlation inequality for $(2, 2)$ or $(3, n)$ settings for $n = 2, 3, 4$, one may conjecture that this is true for general n . Here we prove this.

Theorem 11. *If $\min\{m, n\} \leq 3$, then the inequalities in theorem 8 are all the tight correlation inequalities for (m, n) settings.*

Proof. If $\min\{m, n\} \leq 3$, then $K_{m,n}$ is not contractible to K_5 . Barahona and Mahjoub [4] show that if a graph G is not contractible to K_5 , then all facet-inducing inequalities of $\text{Cut}(G)$ are either the trivial or the cycle inequality. The only facet-inducing cycle inequality of $\text{Cut}(K_{m,n})$ is the CHSH inequality since the only chordless cycle in a complete bipartite graph is a cycle of length four. \square

A similar result was shown in the context of Bell inequalities by Collins and Gisin [11]: a non-negativity inequality $p_{A_i B_j} \geq 0$ and the CHSH inequality are all the inequivalent Bell inequalities for (m, n) settings with $\min\{m, n\} = 2$. As is pointed out in [2], this can also be proved using Barahona and Mahjoub's result.

6.5. Triangular elimination and correlation inequalities

Triangular elimination [2] is a method to convert inequalities valid for the cut polytope $\text{Cut}(K_N)$ of the complete graph to inequalities valid for $\text{Cut}(\nabla K_{m,n})$, which correspond to Bell inequalities via the covariance mapping, preserving their facet-inducing property. In [3], this result is extended to the case of general graphs and in particular the case of bipartite graphs. The inequalities constructed by triangular elimination in the case of bipartite graphs can be regarded as correlation inequalities, as shown by proposition 3 (i).

We describe triangular elimination from $\text{Cut}(K_N)$ to $\text{Cut}(K_{m,n})$ by example. A complete definition and a proof of relevant theorems are stated in [3].

The inequality

$$-x_{A_1 A_2} - x_{A_1 A_3} - x_{A_2 A_3} - x_{B_1 B_2} + \sum_{i=1,2,3} \sum_{j=1,2} x_{A_i B_j} \leq 2 \quad (5)$$

is facet inducing for $\text{Cut}(K_5)$. It is known as the *pentagonal inequality* and is a special case of a hypermetric inequality [13, chapter 28]. This inequality is not a correlation inequality because it depends on the coordinates $x_{A_1 A_2}, x_{A_1 A_3}, x_{A_2 A_3}, x_{B_1 B_2}$, which cannot appear in a correlation inequality. We eliminate a coordinate $x_{A_1 A_2}$ by appending a new node, which we label B_{12} , and adding a triangle inequality. The *triangle inequality* [13, chapter 27] is an inequality in the form $-x_{uv} - x_{uw} - x_{vw} \leq 1$ or in the form $-x_{uv} + x_{uw} + x_{vw} \leq 1$, and also facet inducing for the cut polytope of the complete graph. In this case, adding a triangle inequality $x_{A_1 A_2} + x_{A_1 B_{12}} - x_{A_2 B_{12}} \leq 1$ eliminates the coordinate $x_{A_1 A_2}$. Similarly, we append three more nodes B_{13}, B_{23}, A_{12} and add three appropriate triangle inequalities to eliminate $x_{A_1 A_3}, x_{A_2 A_3}, x_{B_1 B_2}$. This gives the inequality

$$x_{A_1 B_{12}} - x_{A_2 B_{12}} + x_{A_1 B_{13}} - x_{A_3 B_{13}} + x_{A_2 B_{23}} - x_{A_3 B_{23}} + x_{A_{12} B_1} - x_{A_{12} B_2} + \sum_{i=1,2,3} \sum_{j=1,2} x_{A_i B_j} \leq 6, \quad (6)$$

Table 1. The number of facets of $\text{Cut}(K_N)$, the number of tight Bell inequalities obtained as the triangular eliminations of the facets of $\text{Cut}(K_N)$, and the number of tight correlation inequalities in them. Two facets which can be transformed by permutation or switching are considered identical, and two equivalent Bell or correlation inequalities are considered identical. An asterisk (*) indicates the value is a lower bound. The lists of facet inequalities of $\text{Cut}(K_N)$ are obtained from [33]. The number of Bell inequalities is taken from [2].

N	Facets of $\text{Cut}(K_N)$	Tight Bell inequalities	Tight correlation inequalities
3	1	2	1
4	1	2	1
5	2	8	4
6	3	22	10
7	11	323	107
8	147*	40 399*	9159*
9	164 506*	201 374 783*	37 346 094*

or

$$\begin{pmatrix} & (B_1) & (B_2) & (B_{12}) & (B_{13}) & (B_{23}) \\ (A_1) & 1 & 1 & 1 & 1 & 0 \\ (A_2) & 1 & 1 & -1 & 0 & 1 \\ (A_3) & 1 & 1 & 0 & -1 & -1 \\ (A_{12}) & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \leq 6.$$

It is proved in [3] that the inequality constructed in this way is facet inducing for the cut polytope of the complete bipartite graph ($\text{Cut}(K_{4,5})$ in this case).

Tight correlation inequalities constructed in this way can be seen as special cases of Bell inequalities constructed by triangular elimination from K_N to $\nabla K_{m,n}$ which happen to be correlation inequalities. For example, (5) is also facet inducing for $\text{Cut}(K_6)$ with nodes $A_1, A_2, A_3, B_1, B_2, X$ because of the zero-lifting theorem. Applying triangular elimination from $\text{Cut}(K_6)$ to $\text{Cut}(\nabla K_{4,5})$, we construct the same inequality (6).

6.5.1. *Counting correlation inequalities constructed by triangular elimination.* Table 1 shows the number of tight correlation inequalities obtained by triangular elimination from the facet inequalities of $\text{Cut}(K_N)$ for $N \leq 9$. The lists of facet inequalities of $\text{Cut}(K_N)$ are computed by Christof and Reinelt [7] and can be obtained from [33].

6.5.2. *A family of correlation inequalities constructed by triangular elimination.* The following theorem follows from the family of Bell inequalities [2] constructed from hypermetric inequalities [13, section 28] valid for the cut polytope of the complete graph.

Theorem 12. Let $b_{A_1}, \dots, b_{A_s}, b_{B_1}, \dots, b_{B_t}$ be integers such that $\sum_{i=1}^s b_{A_i} + \sum_{j=1}^t b_{B_j} = 1$. Then,

(i) *The inequality*

$$\begin{aligned} & \sum_{1 \leq i \leq s} \sum_{1 \leq j \leq t} b_{A_i} b_{B_j} x_{A_i B_j} + \sum_{1 \leq i < i' \leq s} (b_{A_i} b_{A_{i'}} x_{A_i B_{i'}} - |b_{A_i} b_{A_{i'}}| x_{A_i B_{i'}}) \\ & \quad + \sum_{1 \leq j < j' \leq t} (b_{B_j} b_{B_{j'}} x_{A_{jj'} B_j} - |b_{B_j} b_{B_{j'}}| x_{A_{jj'} B_{j'}}) \\ & \geq \sum_{1 \leq i \leq s} \sum_{1 \leq j \leq t} b_{A_i} b_{B_j} + 2 \sum_{\substack{1 \leq i < i' \leq s \\ b_{A_i} b_{A_{i'}} < 0}} b_{A_i} b_{A_{i'}} + 2 \sum_{\substack{1 \leq j < j' \leq t \\ b_{B_j} b_{B_{j'}} < 0}} b_{B_j} b_{B_{j'}} \end{aligned} \tag{7}$$

is a valid correlation inequality.

(ii) The correlation inequality (7) is facet inducing if one of the following conditions is satisfied.

- (a) For some $l > 1$, the integers $b_{A_1}, \dots, b_{A_s}, b_{B_1}, \dots, b_{B_t}$ contain $l + 1$ entries equal to 1 and l entries equal to -1 , and the other entries (if any) are equal to 0.
- (b) At least 3 and at most $s + t - 3$ entries in $b_{A_1}, \dots, b_{A_s}, b_{B_1}, \dots, b_{B_t}$ are positive, and all the other entries are equal to -1 .

Proof

- (i) If we let $b_X = 0$ in theorem 3.1 of [2] and rewrite the resulting Bell inequality in variables in $x \in \mathbb{R}^{\nabla E_{m,n}}$ by using the covariance mapping, we obtain the inequality (7). What we obtain is an inequality valid for $\text{Cut}(\nabla K_{m,n})$, but it is also an inequality valid for $\text{Cut}(K_{m,n})$ since it does not contain any variables related to the node X .
- (ii) By theorem 3.1 (ii) of [2], the inequality (7) is facet inducing for $\text{Cut}(\nabla K_{m,n})$. From a well-known fact in the theory of polytopes that projecting out unused terms preserves facet-inducing inequalities (see, e.g., lemma 26.5.2 (ii) of [13]), the inequality (7) is facet inducing also for $\text{Cut}(K_{m,n})$. \square

To obtain the inequality (6), we can set $s = 3, t = 2, b_{A_1} = b_{A_2} = b_{A_3} = 1$ and $b_{B_1} = b_{B_2} = -1$ in the inequality 7. Note that this case satisfies the condition (a) in theorem 12 (ii) with $l = 2$.

7. Concluding remarks

We conclude the paper with some open problems. The projection of $\mathcal{Q}_{\text{Cut}}(m, n)$ to $E_{m,n}$ is described directly in terms of the elliptope, but how is the set $\mathcal{Q}_{\text{Cut}}(m, n)$ described? In other words, how close are the sets $\mathcal{Q}_{\text{Cut}}(m, n)$ and $\mathcal{E}(\nabla K_{m,n}) \cap \text{RMet}(\nabla K_{m,n})$?

The upper bound 5.4641 of f_{3322} in quantum correlation experiments obtained by using theorem 6 differs from the known lower bound $9/2$, which is achievable in the two-qubit system. If the upper bound can be improved, it may lead to a refinement of theorem 6. It may be the case that the lower bound $9/2$ is the maximum in the two-qubit system but not in a quantum system with a higher dimension, given that the two-qubit and two-qutrit (three-level) systems are quite different in terms of the ‘strength’ (relevance [11]) of the CHSH inequalities [21].

As we stated in section 4, Tsirelson [35] gives an upper bound on the quantum violation of any correlation inequality by using Grothendieck’s inequality [22]. If Grothendieck’s inequality can be extended to $\nabla K_{m,n}$, we can combine it with theorem 6 to obtain an upper bound of the quantum violation of any Bell inequalities.

Gill [17] asks whether there exists a tight Bell inequality holding for all quantum correlation experiments other than the trivial ones representing the non-negativity of probabilities. Such an inequality corresponds to a facet of $X_{\text{HDB}}(m, n)$ which is valid for $X_{\text{QB}}(m, n)$ but not for $X_{\text{B}}(m, n)$. Asking the same question for tight correlation inequalities corresponds to the question of whether the elliptope $\mathcal{E}(K_{m,n})$ has a facet in common with $\text{Cut}(K_{m,n})$ other than the trivial ones. The facial structure of the elliptope of the complete graph is studied by Laurent and Poljak [24], but we are not aware of any similar results for the bipartite graph.

In connection with the relation between Bell inequalities and quantum games explored by Cleve, Høyer, Toner and Watrous [10], correlation inequalities correspond to XOR games. They pointed out that, from Tsirelson’s theorem, the winning probability of XOR games by

quantum players can be computed efficiently by using semidefinite programming and gave an upper bound on the quantum winning probability by using Grothendieck’s inequality. In a similar way, theorem 6 gives an efficient way to compute an upper bound of the quantum winning probability of general binary games, and if Grothendieck’s inequality can be generalized to $\nabla K_{m,n}$, an analytical upper bound will be given also for binary games.

Extending individual inequalities such as (3) and (4) to classes of inequalities as Collins and Gisin [11] did by introducing the I_{mmvv} inequalities, and researchers in polyhedral combinatorics have done for many classes of inequalities for the cut polytope of the complete graph [13, chapters 27–30], will give better understanding of these correlation inequalities. In addition, it would be useful if we have an analytical bound on the maximum quantum violation for families of correlation inequalities.

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Appendix. Correlation inequalities in (4, 5) settings

We enumerated the facet inequalities of $\text{Cut}(K_{4,5})$ by using cdd [16] (cddlib version 0.94b)⁷. The computation was aborted (seemingly because it ran out the memory), but the partial result shows some of the facets of $\text{Cut}(K_{4,5})$ that are not the zero-lifting of any facets of $\text{Cut}(K_{4,4})$:

$$\begin{pmatrix} & (B_1) & (B_2) & (B_3) & (B_4) & (B_5) \\ (A_1) & 1 & 1 & 1 & 1 & 0 \\ (A_2) & 1 & 1 & -1 & 0 & 1 \\ (A_3) & 1 & 1 & 0 & -1 & -1 \\ (A_4) & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \leq 6,$$

$$\begin{pmatrix} & (B_1) & (B_2) & (B_3) & (B_4) & (B_5) \\ (A_1) & 2 & 1 & 1 & 1 & 1 \\ (A_2) & 0 & 1 & -1 & 1 & -1 \\ (A_3) & 0 & -1 & 1 & 1 & -1 \\ (A_4) & -2 & 1 & 1 & 1 & 1 \end{pmatrix} \leq 8,$$

$$\begin{pmatrix} & (B_1) & (B_2) & (B_3) & (B_4) & (B_5) \\ (A_1) & 2 & 1 & 1 & 1 & 1 \\ (A_2) & -1 & 1 & 2 & 1 & -1 \\ (A_3) & -1 & 2 & 1 & -1 & 1 \\ (A_4) & 0 & 2 & -2 & 1 & -1 \end{pmatrix} \leq 10,$$

⁷ We used the cut cone $\text{CUT}(K_{4,5})$ instead of the cut polytope $\text{Cut}(K_{4,5})$ to reduce the number of facets. This does not essentially change the output since any facet of $\text{Cut}(K_{4,5})$ has a corresponding facet of $\text{CUT}(K_{4,5})$ which is switching equivalent to it (see, e.g., [13, section 26.3.2]).

$$\begin{pmatrix} & (B_1) & (B_2) & (B_3) & (B_4) & (B_5) \\ (A_1) & 1 & 2 & 1 & 1 & -1 \\ (A_2) & 0 & 2 & -1 & -1 & 2 \\ (A_3) & 1 & -1 & 1 & -2 & 1 \\ (A_4) & 0 & -1 & 1 & 2 & 2 \end{pmatrix} \leq 10.$$

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